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MANIFOLDS WITHOUT $\frac{1}{k}$-GEODESICS

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ABSTRACT

We will show that for any positive integer k, there exists a smooth manifold that has no $\frac{1}{k}$-geodesic.

1. Introduction

For compact length spaces, it is known that the classical marked length spectrum may not be continuous with respect to Gromov-Hausdorff limit [1]: there exists a sequence of manifolds M_i, $M_i \to M$ in the Gromov-Hausdorff sense, such that closed geodesics do not persist under this limit. C. Sormani has introduced the $\frac{1}{k}$ **length spectrum** $L_{\frac{1}{k}}(M)$, the set of lengths of $\frac{1}{k}$**-geodesics** in M. A closed geodesic of length l is called a $\frac{1}{k}$**-geodesic** if it is length minimizing on every segment of length $\frac{l}{k}$. Sormani proved that $\frac{1}{k}$**-geodesics** persist under Gromov-Hausdorff limit, which implies that $\frac{1}{k}$ **length spectra** are stable under Gromov-Hausdorff convergence. For discussions about $\frac{1}{k}$ **length spectrum**, see [2].

Sormani showed that every shortest homotopically non-trivial closed geodesic is a $\frac{1}{k}$-geodesic, for all $k \in \mathbb{N}$ [see example below]. Sormani then proposed the following question: does that exist $k \in \mathbb{N}$, such that every smooth, compact, simply connected manifold has a $\frac{1}{k}$-geodesic? We address this question by constructing metrics ρ_k on S^2 for each $k \in \mathbb{N}$, such that (S^2, ρ_k) has no $\frac{1}{k}$-geodesic. That is, we will prove the following theorems.

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THEOREM 1.1: *There exist a metric ρ_2 on S^2 such that (S^2, ρ_2) has no $\frac{1}{2}$-geodesic.*

THEOREM 1.1': *For any fixed k, there exist a metric ρ_k on S^2 such that (S^2, ρ_k) has no $\frac{1}{k}$-geodesic.*

Before we proceed, let us note that these metrics on S^2 have non-negative sectional curvature. Further, $diam(M_k)$ is close to $\sqrt{n^2+1}+1$ and $vol(M_k)$ is close to $(\pi n)/3$, where n is a constant depending on k. In the following, we will construct the manifolds explicitly.

2. Definition

Let M be a smooth manifold. γ: $S^1 \to M$ be a closed geodesic parameterized by arc length and have length l. γ is called $\frac{1}{2}$**-geodesic** if it is distance minimizing on every segment of length $\frac{l}{2}$. Similarly, a $\frac{1}{k}$**-geodesic** is a closed geodesic that is distance minimizing on every segment of length l/k.

Example (Lemma 4.1 of [2]): Suppose that M is not simply connected. Let γ be a shortest homotopically non-trivial closed curve in M. Then γ is a closed geodesic (for instance, see [3]). Let us show that γ is a $\frac{1}{2}$-geodesic. Denote the length of γ by l. Reasoning by contradiction, assume that there are two points p, q on γ that are $l/2$ apart along γ, and that can be connected by a geodesic γ_1 that is shorter than $l/2$. The points p and q divide γ into two geodesics. Each of them can be closed up by adding γ_1. Hence we represented γ as a product of two loops, each of which is shorter than l. Since γ is homotopically non-trivial, so is at least one of these loops. This contradicts our assumption that γ is a shortest homotopically non-trivial loop.

By a **segment** of a geodesic γ we mean the restriction of γ to a closed interval. A $\frac{1}{k}$**-segment** is a segment of length l/k. A **loop** is a finite union of segments that bound a 2 dimensional disc.

3. Construction of the surfaces

Our goal is to show that, for every integer $k \geqslant 2$, there exists a smooth surface M_k that has no $\frac{1}{k}$-geodesic. In our construction, each M_k will be a surface of revolution. First we start with $k = 2$, and then generalize to all k.

THE SURFACE. Consider a curve in ($\mathbb{R}^2$, with the euclidean metric) that consists of a straight line joining (0,1) and (n,0) (n to be determined later), and a straight line from (0,1) to (0,0). These are just two sides of a right triangle. If we revolve this curve about the x-axis, we get a cone K with circular base of radius 1 and height n. Now smoothen the two angles on (0,1) and (n,0) by replacing a small neighborhood of each of the angle with a smooth arc, so that when we revolve it about the x-axis we get a smooth surface. The resulting surface is our M_2. For the sake of simplicity, we create M_2 in the way that the longest parallel (the great parallel) has radius 1. Now, M_2 is diffeomorphic to S^2, and looks like a smoothened cone. Actually, since we alter arbitrarily small neighborhoods of the angles, the surface is **Gromov-Hausdorff** close to K. For instance, such f can be obtained by starting from the midpoint of the hypotenuse. We elongate it by sliding the two ends to sharp angles, followed by a suitable rescaling. Note that M_2 has non-negative sectional curvature (Figure 1).

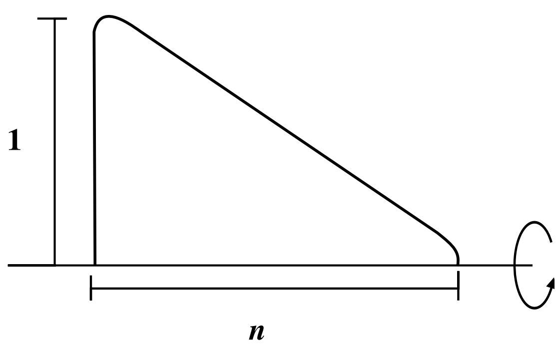

Figure 1. Construction of M_k

The rest of this section is dedicated to proving the following statement:

PROPOSITION 3.1: *With n suitably large, M_2 has no $\frac{1}{2}$-**geodesic**.*

If we can prove Proposition 3.1, using the fact that having a $\frac{1}{k}$-geodesic is a scale invariant concept, we can get the generalized case by rescaling n. To prove the proposition, we will show that all closed geodesics in M_2 are not $\frac{1}{2}$-geodesic. We begin with the following observation:

LEMMA 3.2: *$\frac{1}{2}$-geodesic has no self-intersection.*

Proof. Suppose a closed geodesic γ of length l has self-intersection. Then there exists a segment η with two endpoints coincide, such that η has length $\leqslant l/2$. To see this, suppose γ has at least one self-intersection. Then this self-intersection splits γ into two geodesics, such that the four endpoints coincide at one point. (Think of the figure 8). It is easy to see that one of them has to have length less or equal to $l/2$. Now, any segment of length $l/2$ that contains η cannot be distance minimizing. That is because the two endpoints of this segment can be joined by a shorter path, obtained by deleting η from the segment. ∎

The reason that we consider surfaces of revolution is we can classify all geodesics using **Clairaut's integral** [6]: Given a geodesic, if we denote by r the radius of the parallel which the geodesic intersects with, θ be the angle of intersection. Then the relation

(1) $$r \ \cos\theta = const = c,$$

holds on the whole geodesic.

Using this we have the following

LEMMA 3.3: *No closed geodesic can stay on one side of the great parallel (the longest parallel), i.e. it must intersect the great parallel.*

Proof. Firstly, if γ passes either (n,0) or (0,0), then by Clairaut's integral it has to be a meridian, so it cannot stay on one side. Now suppose on the contrary that the non-meridian geodesic γ stays on one side. By compactness of γ, there exist a shortest and longest parallel (with radius r_1 and r_2), such that γ is tangential to both and lies between them. If $r_1 = r_2$, then γ is a parallel. This cannot happen, since any parallel of this kind is generated by the rotation of a point of the profile curve where the tangent is not parallel to the axis of revolution. None of these parallel can be geodesic [4]. Therefore we must have $r_1 \neq r_2$. This contradicts the Clairaut's integral since in this case, $c = r_1$ and $c = r_2$. ∎

So any geodesic is uniquely determined by the following data: the point of intersection with the great parallel and the angle of intersection α. Now by Clairaut's integral, the angle α determines the constant $c = c_\alpha$. Denote this geodesic $\gamma_\alpha(t)$: $\gamma_\alpha(0)$= the point of intersection with the great parallel.

Let us investigate all closed geodesics in M_2:

Meridians ($\boldsymbol{\alpha = \frac{\pi}{2}}$): Meridians cannot be $\frac{1}{2}$-geodesic if n is large enough. To see this, fix any meridian, its length is approximately $2(n+1)$. Now, pick two points p, q that lie on the same parallel and split the meridian into halves. The distance between p and q is approximately half of the length of the parallel and thus is much shorter than the length of half-meridian.

Great parallel ($\boldsymbol{\alpha = 0}$): The longest parallel (with radius 1) of M_2 cannot be $\frac{1}{2}$-geodesic. Fix any two antipodal points p, q on the great parallel. The distance between p and q along the parallel is π. However p and q can be joined by a path across the base. The length of this path equals approximately the diameter of the great parallel. Which means p and q can be joined by a shorter path. Hence the great parallel is not a $\frac{1}{2}$-geodesic.

Other closed geodesics ($\alpha \in (0, \pi/2)$) require more work. Without loss of generality, we can assume $\gamma'_\alpha(0)$ is pointing into the cone. Let $r_\alpha(t)$ be the radius of parallel intersecting γ_α at $\gamma_\alpha(t)$, and $\theta_\alpha(t)$ be angle of intersection. Observe that when $r_\alpha(t_\alpha) = c_\alpha$, for some $t_\alpha \in [0, l]$, γ_α is tangential to the parallel, and then it will start to return [6]. Denote by R_α the parallel where γ_α start to turn back.

Definition 3.4: For each $\alpha \in [0, \pi/2)$, define the total rotation $T_\alpha(t),\ t \in [0, l]$ to be the net (oriented) angle of rotation of γ_α about the axis of revolution from $\gamma_\alpha(0)$ to $\gamma_\alpha(t)$.

Example: When α=0, γ_α is just the great parallel, Therefore $T_\alpha(t) = \pm t$ (depending on the orientation chosen).

First, for any $\alpha \neq \pi/2$, $|T_\alpha(t)|$ is a monotonic increasing function. This is equivalent to saying that any non-meridian geodesic γ rotates only in one direction. To prove this claim, assume on the contrary that γ changes rotational direction at some point. Then at this point, γ should be tangential to a meridian. By the uniqueness of geodesics (in a smooth manifold, a point and a vector uniquely determine a geodesic), γ should coincide with a meridian. This contradicts the assumption that γ is a non-meridian.

Now recall that $\gamma_\alpha(t_\alpha)$ is the point when γ_α turns back, we have the following

Lemma 3.5: *If $|T_\alpha(t_\alpha)| > \pi$, then γ_α has self-intersection.*

Proof. We know from Clairaut's integral that γ_α cannot touch the great parallel. So if $|T_\alpha(t_\alpha)| > \pi$, the total rotation of γ_α in the cone area is strictly greater than 2π, which implies there is a self-intersection. ∎

We are now ready to list all the remaining geodesics in M_2, to simplify our argument, let us divide M_2 into four areas. Recall that in our construction, we smoothen 2 corners of the generating curve. Therefore, when we revolve it: There is a curved cap at the tip (the cap), a thin curved belt around the great parallel (the belt), a flat disc at the bottom (the disc) and the long cone (the cone) [Figure 2]. Only the cap and the belt have non-zero curvature.

The remaining geodesics can be divided into three types:

a) Geodesics that never leave the belt before returning to the great parallel.
b) Geodesics that enter the cap.
c) Geodesics that enter the cone but miss the cap.

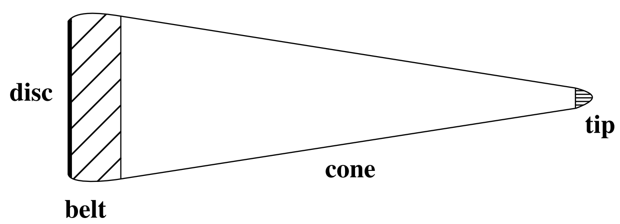

Figure 2. Four areas of M_2

There are two parallels; the one seperating the belt and the cone and the one seperating the cone and the cap. Denote these two parallels by R' and R'' respectively. Now since in constructing M_2, the belt and the cap can be arbitrarily thin. We can choose them to be so thin that for some chosen α' and α'' so that α', $(\pi/2 - \alpha'') \ll \pi/2$, we have $R' = R_{\alpha'}$ and $R'' = R_{\alpha''}$. To make the following arguments simpler, we also dilate M_2 proportionally so that $R_{\alpha'}$ has length 1. There is no impact on all previous arguments because they held on all our manifolds regardless of scaling and the region of smoothing. Also, we denote the distance between $R_{\alpha'}$ and the great parallel by ϵ, diameter of the cap be ϵ', where $\epsilon, \epsilon' \ll 1$.

The three cases of geodesics are equivalent to:

a) $\alpha \in (0, \alpha')$
b) $\alpha \in [\alpha'', \pi/2)$

c) $\alpha \in [\alpha', \alpha'')$

Case a): If γ_α wraps around M_2 twice or more, its winding number about the base's center is greater than 2, so γ_α has self-intersection. Hence by lemma 3.2, γ_α is not $\frac{1}{2}$-geodesic. If γ_α only wrap around M_2 once, then it enters each side of the great parallel once. Hence γ_α's length is within $2\pi \pm 10\epsilon$. Therefore γ_α is similar to the great parallel: any two points p, q dividing γ_α into halves can be joined by a path of length $\leq 2 + 10\epsilon$. This is a shorter path. Therefore we conclude that all geodesics in this case are not $\frac{1}{2}$-geodesic.

Case b): Now, since γ_α connects the great parallel and some point in the cap, γ_α is of length at least $(2n - \epsilon')$. Then it is just like the meridian case: find two points which are $\frac{2n-\epsilon'}{2}$ apart and lie on the same parallel. When n is large the half-parallel is a shorter path. Hence no geodesic in case b can be $\frac{1}{2}$-geodesic.

Case c): If γ_α enters the cone, then it must cross the parallel $R_{\alpha'}$. So there is an angle of intersection $\tilde{\alpha}$ between γ_α and $R_{\alpha'}$. Define $T(\tilde{\alpha})$, the **first return rotation** to be the total rotation of γ_α from $R_{\alpha'}$ and the point when it first hit $R_{\alpha'}$ again (Figure 3).

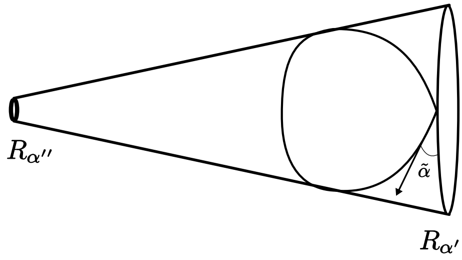

Figure 3. $T(\tilde{\alpha}) = 2\pi$

We need the following

Lemma 3.6: *$T(\tilde{\alpha})$ is monotonic increasing in $\tilde{\alpha}$ for all geodesics in case c.*

Proof. Consider the universal cover of the cone. Construct it by starting with an annulus, cut through one radius. Then take another copy of the same thing and glue the left side of the cut from the first copy to the right side of the second copy. Continuing infinitely we get the universal cover. It looks like a infinite spiral and is a topological infinite strip. A fundamental domain is a sector (Figure 4).

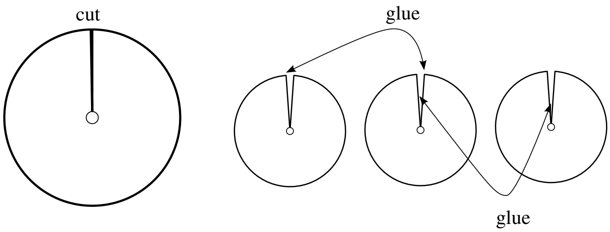

Figure 4. The universal cover of the cone

Now this is a development of the cone area, any geodesic segment is a straight line. Also, $\tilde{\alpha}$ is given by the angle of intersection with the outer circle. It is now easy to see that $T(\tilde{\alpha})$ is monotonic increasing in $\tilde{\alpha}$: Since we assume that $R_{\alpha'}$ has length 1, $T(\tilde{\alpha})$ is the length of the arc corresponding to the chord given by γ_α (Figure 5). ∎

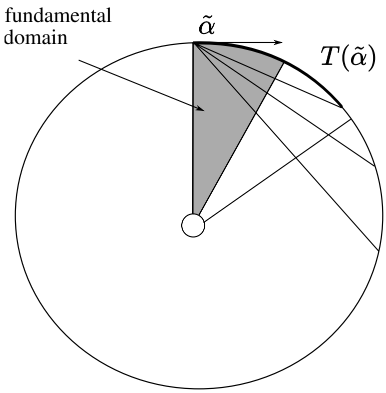

Figure 5. $T(\tilde{\alpha})$ is monotonic increasing

Finally, we claim that for any fix $\zeta \geq \epsilon$. When n is large enough, any γ_α not contained in the ζ-neighborhood of the great parallel has self-intersection. To see this, consider the fundamental domain (with arc length $l(R_{\alpha'}) = 1$). A chord connecting two end points of the arc is a geodesic γ_α with $T(\tilde{\alpha}) = 2\pi$. Denote by L the distance between $R_{\alpha'}$ and γ_α. Elementary calculation shows that $L = n\big(1 - \sqrt{1 - \sin^2 \frac{1}{2n}}\big) \to 0$ as $n \to \infty$ (Figure 6). So when n is large enough such that $L = \zeta$, the geodesic that turns back exactly at the boundary of the ζ-neighborhood gives $T(\tilde{\alpha}) = 2\pi$, hence it has self-intersection. Together with Lemma 3.6., when γ_α is not contained in the ζ-neighborhood of the great parallel, it has self-intersection. Therefore by Lemma 3.2, such geodesic cannot be $\frac{1}{2}$-geodesic.

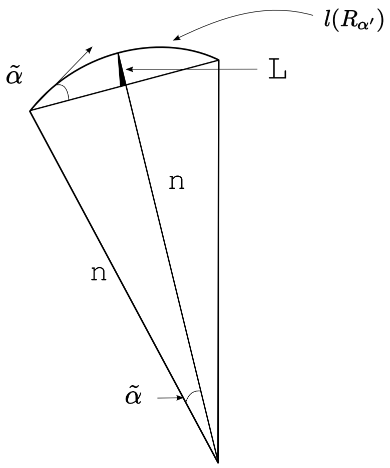

Figure 6. $L \to 0$ as $n \to \infty$

Now, the remaining geodesics are those that sit inside the ζ-neighborhood of the great parallel. Take $\zeta \ll 1$, this is similar to the case where the geodesics is contained in the curved belt: any two points p, q dividing γ_α into halves can be joined by a shorter path through the disc.

So if we choose n large enough such that all the previous criteria are met. Then M_2 has no $\frac{1}{2}$-geodesic and we finish the proof of Proposition 3.1 and thus Theorem 1.1.

4. When $k \geq 3$

Now we move to prove the general case. The construction of M_k is similar to that of M_2, except that we have to use larger n, thinner belt and smaller cap.

Proposition 4.1: *For any fixed k, M_k has no $\frac{1}{k}$-geodesic.*

Similar to what we have done before, we will exhibit all possible geodesics. First off, any closed geodesic γ must intersect the great parallel (Lemma 3.3). So as before we can use the angle of intersection α to characterize the geodesics. In what followes we still assume that γ has length l.

Meridians: Meridians are not $\frac{1}{k}$-geodesic if n is large enough. Again, find two points p, q near the tip that contain a $1/k$ segment and lie on the same parallel. n being large implies l/k is much larger than the length of any parallel. Therefore there is a shorter path joining p, q.

Great parallel: The great parallel has length 2π. Any two points p, q that contain a $1/k$ segment ($(2\pi)/k$ long) of the great parallel can be joined by a shorter path through the base. This is a chord on the disc plus some small error. For any k, we can make the width of the smoothing to be narrow enough so that the error term is much smaller than $(2\pi)/k$. Therefore the great parallel is not a $\frac{1}{k}$-geodesic.

Other geodesics: Again, these geodesics can be categorized into 3 types: stays in the belt, goes into the cap and goes into the cone but not the cap.

1) In the belt: If the geodesic wraps around once, then it is similar to the case of the great parallel: p, q can be joined by a shorter path close to a chord of the great parallel. If the geodesic wraps around m times, then for p, q bounding a $\frac{1}{k}$ segment, they are apart by approximately $(2m\pi)/k > (2\pi)/k$. Again, p, q can be joined by a shorter path through the disc.

2) Into the cap: Similar to the case of $k = 2$, any geodesic that runs into the cap has length at least $2n - \epsilon'$ for some small ϵ'. We can find p, q near the tip. Such that p, q bound a $1/k$ segment ($\frac{2n-\epsilon'}{k}$ long) of the geodesic, and lie on the same parallel. Then p, q can be joined by a path close to a half-parallel which is a shorter path.

3) Geodesics that run into the cone but miss the cap: Since $k \geq 3$, Lemma 3.2 no longer applies here. However, we have the following lemma:

LEMMA 4.2: *For any γ_α in case 3. If γ_α has $(k+1)$ self-intersections in the cone area. Then γ_α is not a $\frac{1}{k}$-geodesic.*

Proof. Suppose γ_α has $(k+1)$ self-intersections in the cone area. Recall that by Clairaut's integral, any geodesic of this form is symmetric about the meridian that contains the point where the geodesic starts to turn back. The self-intersections split γ_α into at least $(2k+1)$ segments. Let us label the corresponding segments 1, 2, 2', etc. (Figure 7). Notice that segment 1 forms a loop, segments 2 and 2' form another loop and so on. There are altogether k loops of this kind in the cone area.

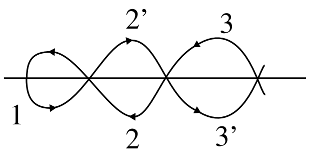

Figure 7. A geodesic in case 3

Now we consider the universal cover again. Since segment 1 is the only one which is orthogonal to a meridian. It has to be strictly shorter than $length(segment\ i) + length(segment\ i')$ for $2 \leq i \leq k$ (Figure 8). That means segment 1 is the shortest loop among the k loops in the cone area. Which implies $length(segment\ 1) < l/k$. Any $1/k$ segment of γ_α containing segment 1 cannot be shortest path. Since we can connect the two endpoints by a shorter path if we jump segment 1 at the point of intersection. ∎

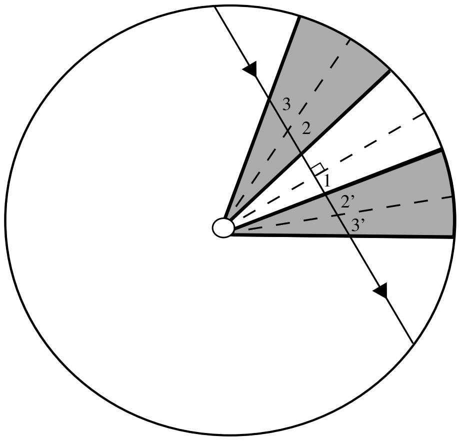

Figure 8. Segment 1 has length $\leq l/k$.

Now, given any fixed ζ, $\epsilon < \zeta \ll 1/k$. Using the same argument as $k = 2$: When n is large enough, the geodesic in figure 8 crosses at least $(k+1)$ fundamental domains, therefore $T(\tilde{\alpha}) > 2(k+1)\pi$ for all γ_α not contained inside the ζ-neighborhood of the great parallel. This implies that γ_α has $(k+1)$

self-intersections and by Lemma 4.2, γ_α is not $\frac{1}{k}$-geodesic. If γ_α is contained inside the ζ-neighborhood, then $\zeta \ll \frac{1}{k}$ implies γ_α is similar to those in case 1, hence it cannot be $\frac{1}{k}$-geodesic.

So for n large enough, M_k has no $\frac{1}{k}$-geodesic.

We have thus completed the proof of Proposition 4.1 and therefore Theorem 1.1'.

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